Response functions in multicomponent Luttinger liquids

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We derive an analytic expression for the zero temperature Fourier transform of the density-density correlation function of a multicomponent Luttinger liquid with different velocities. By employing Schwinger identity and a generalized Feynman identity exact integral expressions are derived, and approximate analytical forms are given for frequencies close to each component singularity. We find power-like singularities and compute the corresponding exponents. Numerical results are shown for $N = 3$ components and implications for experiments on cold atoms are discussed.

I. INTRODUCTION

It is well known that as a result of spin-charge separation, interacting one-dimensional spin-1/2 fermions with repulsive interaction at incommensurate filling form a two-component Luttinger liquid (LL). Multicomponent LL with more than two components can be obtained in fermionic systems with repulsive interaction[1–3]. Possible realization of such systems are provided by multichannel quantum wires[4], carbon nanotubes[5,6] and biased bilayer graphene[7]. Interaction of acoustic phonons with spin-1/2 fermions in one dimension can also give rise to a multicomponent LL.[8,9] In Mott insulating materials, a multicomponent LL can be formed in spin-orbital chains[10–18] and spin tubes[19] under the effect of an applied magnetic field[20,21] and in spin-1 chains with biquadratic interactions[22,23]. More recently, atom trapping technology has permitted the realization of Bose-Fermi mixtures,[24–27] as well as degenerate gases with internal degrees of freedom.[28,29] In the latter case, it has been suggested theoretically that these systems could realize SU(N) spin systems[30–33] in low dimensions. In parallel, techniques for trapping atoms in one dimension have been developed[34–41]. In Bose-Fermi mixtures trapped in one dimension, a multicomponent LL behavior is expected[42–56]. Similarly, multicomponent systems with repulsive interactions are also expected to exhibit in one dimension a multicomponent LL behavior[57,58] The real space correlation function of the multicomponent Luttinger liquid can be readily obtained[8,50,59]. However, the majority of experimental observables are actually Fourier transform (FT) of these correlation functions and thus the FT of the multicomponent LL has to be obtained. Because of the branch cut structure of the correlation functions in a LL, this is a non-trivial task. In the single component case, the calculation can be done in closed form[60–62]. For two component systems, only the exponents of the power law singularities could be predicted.[63–65] Recently a closed form of the $2k_F$ component of the density-density response function in a two-component LL (i.e. the spin-1/2 case with different charge and spin velocites) at zero temperature was obtained[66] in terms of Appell hypergeometric functions.[67,68] Such an expression permits the description of the crossovers between the different power law singularities of the response functions. In the present manuscript, we derive an exact expression for the FT of the density-density correlation function in the general case of a multicomponent LL with different velocities for the modes. We show that in this general case, the FT of the Matsubara correlation functions are expressed in terms of the Srivastava-Daoust generalized hypergeometric functions. We give give the full analytical continuation of the correlation functions to real frequencies, recovering the leading power-law singularities and describing the various crossovers between them.

The paper is so organized. In Sec. II we give the Hamiltonian of a general multicomponent system and derive the general expression for the Matsubara correlation functions at zero temperature. In the subsections II A and II B we derive two exact integral representations of the Fourier transforms of the Matsubara correlation functions by means of a Schwinger identity and a Feynman identity, respectively. The integral representation obtained from the Schwinger identity is used to predict the exponents of the power-law singularities. In Sec.II C, starting from the integral representation derived from the Feynman identity, we obtain the analytic continuation of the Matsubara correlator in various cases and give the asymptotic expression close to the singular points. Finally, we give some conclusions in Sec.III.

II. MODEL

In the present paper, we wish to consider the case of a general multicomponent system. The continuum Hamiltonian is:

$$\begin{aligned} H &= \sum_{a=1}^{N} \int dx \left[-\psi_a^\dagger \frac{\hbar^2}{2M_a} \partial_x^2 \psi_a \right] \\ &+ \sum_{1\leq a<b\leq N} \int dx dx' V_{ab}(x-x')\rho_a(x)\rho_a(x'), \end{aligned} \tag{1}$$

where ψ_a annihilates a particle of type a, $\rho_a(x) = \psi_a^\dagger \psi_a$ is the corresponding particle density, and M_a the corresponding mass. The interaction of particles of type a with particles of type b is V_{ab}. The particles may be either bosons or fermions. The Hamiltonian (1) can be bosonized[69] and its expression reads:

$$H = \sum_{a,b} \int \frac{dx}{2\pi} \left[(\pi \Pi_a) M_{ab} (\pi \Pi_b) + (\partial_x \phi_a) N_{ab} (\partial_x \phi_b) \right], \tag{2}$$

where $[\phi_a(x), \Pi_b(x')] = i\delta_{ab}\delta(x-x')$, and the matrices M and N can be obtained respectively from the variation of the ground state energy with change of boundary conditions and particle numbers.[69] The Hamiltonian (2) can be diagonalized, and the equal time correlations can then be obtained.[69] We wish to calculate the Matsubara time dependent correlation functions. In the case of density correlations, since the density is expressed as:

$$\rho_a(x) = -\frac{1}{\pi}\partial_x \phi_a + \sum_{m=-\infty}^{+\infty} A_m \cos[2m(\phi_a(x) - \pi\rho_a^0 x)], \tag{3}$$

We will need to calculate correlation functions of the form $\langle T_\tau e^{im\phi_a(x,\tau)} e^{-im\phi_a(0,0)} \rangle$, where T_τ is the Matsubara time ordered operator. If we have bosonic particles, the bosonized form of the annihilation operator being:

$$\psi_a(x) = e^{i\theta_a(x)} \left[\sum_m B_m e^{i2m[\phi_a(x) - \pi\rho_a^0 x]} \right], \tag{4}$$

where $\nabla\theta_a = \pi\Pi_a$, the leading term in the single-particle Green's function is proportional to $\langle T_\tau e^{i\theta_a(x,\tau)} e^{-i\theta_a(x,\tau)} \rangle$. Because of the duality transformation[69] $M \leftrightarrow N$ and $\theta_a \leftrightarrow \phi_a$, it is sufficient to calculate the correlation functions of the form $\langle T_\tau e^{i\sum_n \alpha_n \phi_n(x,\tau)} e^{-i\sum_n \alpha_n \phi_n(0,0)} \rangle$ and use the duality transformation to obtain the correlation functions $\langle T_\tau e^{i\sum_n \alpha_n \theta_n(x,\tau)} e^{-i\sum_n \alpha_n \theta_n(0,0)} \rangle$. We have:

$$\langle T_\tau e^{i\sum_n \alpha_n \phi_n(x,\tau)} e^{-i\sum_n \alpha_n \phi_n(0,0)} \rangle = \exp\left[-\sum_{n<m} \alpha_n \alpha_m G_{nm}(x,\tau) \right], \tag{5}$$

with[69],

$$G_{nm}(x,\tau) = \pi \sum_{\omega_n} \int \frac{dq}{2\pi} e^{-|q|\alpha} (1 - e^{i(qx - \omega_n \tau)}) [(\omega_n^2 + (MN)q^2)^{-1} M]_{nm}. \tag{6}$$

Let us introduce the projection operator P_λ that projects on the eigenspace of eigenvalue u_λ^2 of the matrix MN to rewrite the matrix $(\omega_n^2 + (MN)q^2)^{-1}$ as:

$$(\omega_n^2 + (MN)q^2)^{-1} = \sum_n \frac{P_n}{\omega_n^2 + u_n^2 q^2} \tag{7}$$

In the limit of $\beta \to \infty$, we have:

$$\int \frac{d\omega}{2\pi} (\omega^2 + (MN)q^2)^{-1} (1 - e^{i(qx - \omega\tau)}) e^{-|q|\alpha} = \sum_\lambda \frac{P_\lambda}{2u_\lambda |q|} (1 - e^{iqx - u_\lambda |q\tau|}) e^{-|q|\alpha}, \tag{8}$$

So that:

$$G(x,\tau) = \frac{1}{4}\sum_\lambda \frac{P_\lambda}{u_\lambda} M \ln\left(\frac{x^2+(u_\lambda|\tau|+\alpha)^2}{\alpha^2}\right), \tag{9}$$

and thus at zero temperature:

$$\langle T_\tau e^{i\sum_n \alpha_n\phi_n(x,\tau)} e^{-i\sum_n \alpha_n\phi_n(0,0)}\rangle = \prod_\lambda \left(\frac{\alpha^2}{x^2+(u_\lambda|\tau|+\alpha)^2}\right)^{\eta_\lambda}, \tag{10}$$

with:

$$\eta_\lambda = \sum_{a,b} \alpha_a \frac{(P_\lambda M)_{ab}}{u_\lambda}\alpha_b. \tag{11}$$

In the case of the θ correlation function, one obtains a formula analogous to (10), with the exponents η_n replaced by $\bar{\eta}_n$ where:

$$\bar{\eta}_\lambda = \sum_{a,b} \alpha_a \frac{(Q_\lambda N)_{ab}}{u_\lambda}\alpha_b, \tag{12}$$

Q_λ being the projector on the eigenstate of NM having the eigenvalue u_λ^2. We note that for $\tau = 0$, using Eqs. (10)–(11), we recover the expression of the exponents of equal-time correlations from[69]. These formulas are particularly useful when implementing numerical methods like DMRG and exact diagonalization as one could use the following results to predict the response function from ground state energy computations. Knowing the Matsubara time ordered Green's function, Eq. (10), we wish to obtain the corresponding Matsubara response function:

$$\chi^M(q,i\omega_n) = \int dx d\tau e^{i(qx-\omega_n\tau)} \langle T_\tau e^{i\sum_n \alpha_n\phi_n(x,\tau)} e^{-i\sum_n \alpha_n\phi_n(0,0)}\rangle, \tag{13}$$

and the retardated response function $\chi(q,\omega) = \chi^M(q,i\omega_n \to \omega + i0)$. In the following, we use two complementary approaches, the first one based on the Schwinger identity[70] in Sec. II A, that will allow us to predict the singularities of the retardated response function, and the second one based on the Feynman identity[70] that will allow us to make a connection with the results for the two-component case[66]. We will assume that we can take the limit $\alpha \to 0$ in the integrals, i. e. $\eta = \sum \eta_n < 1$.

Let us remind that for the density-density correlation function its FT is related to the scattering cross section σ of light at a frequency ω and angle Ω incident on a sample, by the relation

$$\frac{d^2\sigma}{d\omega d\Omega} \propto S(q,\omega) = \mathrm{Sign}(\omega)\mathfrak{Im}\chi(q,\omega) \tag{14}$$

where $S(q,\omega)$ is the dynamic structure factor. This quantity is accessible e.g. by means of inelastic neutron/light scattering when spin/density fluctuations are induced in the system and their subsequent relaxation is measured.

A. Schwinger identity

The Schwinger identity is[70]:

$$\frac{1}{(x^2+(u\tau)^2)^\alpha} = \int_0^{+\infty} \frac{d\lambda}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda(x^2+(u\tau)^2)} \tag{15}$$

In the multicomponent case, it allows us to rewrite the expression (10) as:

$$\prod_n \left(\frac{\alpha^2}{x^2+(u_n|\tau|+\alpha)^2}\right)^{\eta_n} = \int \prod_n \frac{d\lambda_n \lambda_n^{\eta_n-1}}{\Gamma(\eta_n)} \exp\left[-x^2\left(\sum_n \lambda_n\right) - \tau^2\left(\sum_n \lambda_n u_n^2\right)\right] \tag{16}$$

The Fourier transformation in (13) reduces to a Gaussian integral, and we find:

$$\chi(q,i\omega_n) = \pi\alpha^{2\sum\eta_n} \int \prod_n \frac{d\lambda_n \lambda_n^{\eta_n-1}}{\Gamma(\eta_n)} \frac{e^{-\frac{1}{4}\left[\frac{q^2}{(\sum_n\lambda_n)} + \frac{\omega_n^2}{(\sum_n \lambda_n u_n^2)}\right]}}{\sqrt{(\sum_n \lambda_n)(\sum_n \lambda_n u_n^2)}} \tag{17}$$

With the change of variables:

$$\lambda_j = \lambda_1 \mu_{j-1} (j = 2, \ldots, N), \tag{18}$$

we rewrite (17) as:

$$\begin{aligned}\chi(q, i\omega_n) &= \pi\alpha^{2\sum \eta_n} \int \frac{d\lambda_1}{\Gamma(\eta_1)} \lambda_1^{\sum \eta_n - 2} \\ &\int \prod_{j=1}^{N-1} \frac{d\mu_j \mu_j^{\eta_{j+1}-1}}{\Gamma(\eta_{j+1})} \frac{e^{-\frac{1}{4\lambda_1}\left[\frac{q^2}{1+\sum_{j=1}^{N-1}\mu_j} + \frac{\omega^2}{u_1^2 + \sum_{j=1}^{N-1}\mu_j u_{j+1}^2}\right]}}{\sqrt{\left(1+\sum_{j=1}^{N-1}\mu_j\right)\left(u_1^2 + \sum_{j=1}^{N-1}\mu_j u_{j+1}^2\right)}}\end{aligned} \tag{19}$$

With the change of variable $\lambda_1 = \mu^{-1}$, the integration over λ_1 can be done in closed form. We find, provided that $\sum \eta_n < 1$:

$$\chi(q, i\omega_n) = \pi\alpha^{2\sum \eta_n} \frac{\Gamma(1 - \sum_j \eta_j)}{\prod_j \Gamma(\eta_j)} \int \prod_{j=1}^{N-1} d\mu_j \mu_j^{\eta_{j+1}-1} \frac{\left[\frac{q^2}{1+\sum_{j=1}^{N-1}\mu_j} + \frac{\omega^2}{u_1^2 + \sum_{j=1}^{N-1}\mu_j u_{j+1}^2}\right]^{\sum \eta_n - 1}}{\sqrt{\left(1+\sum_{j=1}^{N-1}\mu_j\right)\left(u_1^2 + \sum_{j=1}^{N-1}\mu_j u_{j+1}^2\right)}} \tag{20}$$

We can perform the analytic continuation on (17) by substituting $i\omega_n \to \omega + i0$. When $\omega \simeq u_1 q$, we have to consider the integral:

$$\int \prod_{j=1}^{N-1} d\mu_j \mu_j^{\eta_{j+1}-1} \left[u_1^2 q^2 - \omega^2 + \sum_{j=1}^{N-1} \mu_j (u_{j+1}^2 q^2 - \omega^2)\right]^{\sum \eta_n - 1} \tag{21}$$

With the change of variables $\mu_j = (u_1^2 q^2 - \omega^2)\xi_j$, we finally find that for $|\omega| \to u_1 q$,

$$\chi(q, \omega) \sim |(u_1 q)^2 - \omega^2|^{2(\sum \eta_n) - \eta_1 - 1}, \tag{22}$$

provided $2(\sum \eta_n) - \eta_1 - 1 < 0$. In the general case, we expect to find $\chi(q, \omega \simeq u_j q) \sim\sim |(u_j q)^2 - \omega^2|^{2(\sum_n \eta_n) - \eta_j - 1}$ when $2(\sum \eta_n) - \eta_j - 1 < 0$. We note that $\eta = \sum_n \eta_n$ is the exponent of the equal time correlation functions.[69] The exponents of the singularities satisfy a "sum rule":

$$\sum_{n=1}^{N} (2\eta - \eta_n - 1) = (2N - 1)\eta - N \tag{23}$$

B. Feynman identity

Using the identity from[70] (Appendix B), we can rewrite the correlation function (10) as a multiple integral:

$$\prod_{j=1}^{N} \frac{1}{(x^2 + u_n^2 \tau^2)^{\eta_n}} = \frac{\Gamma(\sum_1^2 \eta_n)}{\prod_{j=1}^{N} \Gamma(\eta_j)} \int \prod_{j=1}^{N} dw_j w_j^{\eta_j - 1} \delta(1 - \sum_{j=1}^{N} w_j) \left(x^2 + \sum_{j=1}^{N} w_j u_j^2 \tau^2\right)^{-(\sum_{j=1}^{N} \eta_j)} \tag{24}$$

The Matsubara response function (13) is then found from the integral(11.4.16) of Ref. 71:

$$\begin{aligned}&\int dx d\tau e^{i(qx - \omega_n \tau)} \left(x^2 + \sum_{j=1}^{N} w_j u_j^2 \tau^2\right)^{-\eta} \\ &= \frac{2^{2(1-\eta)} \pi \Gamma(1-\eta)}{\Gamma(\eta)\sqrt{\sum_{j=1}^{N} w_j u_j^2}} \left(q^2 + \frac{\omega_n^2}{\sum_{j=1}^{N} w_j u_j^2}\right)^{\eta - 1},\end{aligned} \tag{25}$$

where $1/4 < \eta < 1$ as:

$$\chi(q, i\omega_n) = \frac{\Gamma(1-\eta)}{\prod_{j=1}^{N}\Gamma(\eta_j)} \int \prod_{j=1}^{N} dw_j w_j^{\eta_j - 1} \delta(1 - \sum_{j=1}^{N} w_j) \frac{2^{2(1-\eta)}\alpha^{2\eta}\pi}{\left(\sum_{j=1}^{N} w_j u_j^2\right)^{\eta-1/2}} \left(\omega_n^2 + \sum_{j=1}^{N} w_j u_j^2 q^2\right)^{\eta-1}, \tag{26}$$

where we have defined $\eta = \sum_j \eta_j$. For the case of $\eta = 1/2$, the integral (26) can be expressed as a Lauricella function F_D, which actually reduces to a simple product. The result is simply:

$$\chi(q, i\omega_n) = 2\pi\alpha \prod_{j=1}^{N} (\omega^2 + u_j^2 q^2)^{-\eta_j}. \tag{27}$$

This result generalizes the one obtained for the two-component case in Ref. 66.

When $\eta > 1/2$, we can use again the Feynman identity[70] to write:

$$\frac{1}{\left(\frac{\omega^2}{q^2} + \sum_j u_j^2 w_j\right)^{1-\eta} \left(\sum_j u_j^2 w_j\right)^{\eta-1/2}} = \frac{\Gamma(1/2)}{\Gamma(\eta - 1/2)\Gamma(1-\eta)} \int_0^1 ds s^{-\eta}(1-s)^{\eta-3/2} \left(s\frac{\omega^2}{q^2} + \sum_j u_j^2 w_j\right)^{-1/2}, \tag{28}$$

and find:

$$\chi^{(M)}(q, i\omega_n) = \frac{\pi 2^{2(1-\eta)}\alpha^2\Gamma(1/2)}{\Gamma(\eta-1/2)\Gamma(\eta)} (q\alpha)^{2(\eta-1)} \int_0^1 ds \quad \frac{s^{-\eta}(1-s)^{\eta-3/2}}{\left(\frac{\omega^2}{q^2}s + u_N^2\right)^{1/2}} \times F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2 + s\frac{\omega^2}{q^2}}, \ldots, \frac{u_N^2 - u_{N-1}^2}{u_N^2 + s\frac{\omega^2}{q^2}}\right), \tag{29}$$

where the label (M) stands for the Matsubara correlation function and $F_D^{(N-1)}$ is a Lauricella hypergeometric function of $N-1$ variables. The integral (29) can be expressed in closed form[72] using Srivastava-Daoust generalized hypergeometric series[73] of N variables as:

$$\chi^{(M)}(q, i\omega_n) = F(\eta)(q\alpha)^{2(\eta-1)} F^{1;1,\ldots,1,1}_{1;0,\ldots,0,1}\left[\begin{array}{l} (1/2; 1, \ldots, 1, 1): \ \eta_1, \ \ldots, \eta_{N-1}, \ 1-\eta \\ (\eta; 1, \ldots, 1, 0): \ -, \ \ldots, \ -, \ 1 \end{array}; 1 - \frac{u_1^2}{u_N^2}, \ldots, 1 - \frac{u_{N-1}^2}{u_N^2}, -\frac{\omega^2}{u_N^2 q^2}\right], \tag{30}$$

where $F(\eta) = \frac{\pi 2^{2(1-\eta)}\alpha^2\Gamma(1-\eta))}{\Gamma(\eta)u_N}$. When $N = 2$, the Srivastava-Daoust hypergeometric series reduces to an Appell F_2 hypergeometric function of 2 variables. Using the identity (16.16.3) from [68], this function is seen to reduce to the F_1 Appell function expression given in Ref. [66]. Since we are not aware of any study of the analytic continuation of the Srivastava-Daoust hypergeometric series outside their circle of convergence, we will not pursue with Eq. (30). Instead, we will consider directly the analytic continuation of the integral (29). Introducing the new variable:

$$t = \frac{\omega_n^2 + u_N^2 q^2}{s\omega_n^2 + u_N^2 q^2} s, \tag{31}$$

the integral (29) is rewritten:

$$\chi^{(M)}(q, i\omega_n) = \frac{\pi 2^{2(1-\eta)}\alpha^2\Gamma(1/2)}{\Gamma(\eta-1/2)\Gamma(\eta)u_N} \left(\frac{\omega_n^2\alpha^2}{u_N^2} + (q\alpha)^2\right)^{(\eta-1)} \int_0^1 dt t^{-\eta}(1-t)^{\eta-3/2} \times F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2}\left(1 - \frac{\omega_n^2 t}{\omega_n^2 + u_N^2 q^2}\right), \ldots, ; \frac{u_N^2 - u_{N-1}^2}{u_N^2}\left(1 - \frac{\omega_n^2 t}{\omega_n^2 + u_N^2 q^2}\right)\right), \tag{32}$$

In the case $N = 2$, the Lauricella function in the integral (32) reduced to a ${}_2F_1$ Gauss hypergeometric function. Doing the integral gives a F_1 Appell hypergeometric function[67]. With $N = 3$, the Lauricella function reduces to a F_1 Appell function.

In the case of $\eta < 1/2$, the integral (29) is divergent. In order to obtain a convergent integrals, we write in Eq. (26):

$$\frac{\left(\omega^2 + q^2 \sum_j u_j^2 w_j\right)^{\eta-1}}{\left(\sum_j u_j^2 w_j\right)^{\eta-1/2}} = \sum_{j=1}^{N} \frac{u_j^2 w_j}{\left(\omega^2 + q^2 \sum_j u_j^2 w_j\right)^{1-\eta} \left(\sum_j u_j^2 w_j\right)^{\eta+1/2}}, \tag{33}$$

and apply the Feynman identity to each term in the sum. We thus find:

$$\begin{aligned} \chi(q, i\omega_n) &= \pi\alpha^{2\eta} q^{2\eta-2} 2^{2(1-\eta)} \frac{\Gamma(3/2)}{\Gamma(\eta+1/2)\Gamma(\eta+1)} \\ &\times \int_0^1 ds \frac{s^{-\eta}(1-s)^{\eta-1/2}}{\left(s\frac{\omega^2}{q^2} + u_N^2\right)^{3/2}} \left[\sum_{\ell=1}^{N} \eta_\ell u_\ell^2 F_D^{(N-1)} \left(\frac{3}{2}; \{\eta_j + \delta_{j\ell}\}_{1\le j\le N-1}; \eta+1; \frac{u_N^2 - u_1^2}{s\frac{\omega^2}{q^2} + u_N^2}, \ldots, \frac{u_N^2 - u_1^2}{s\frac{\omega^2}{q^2} + u_N^2}\right)\right] \end{aligned} \tag{34}$$

Each term in the sum is then expressible with Srivastava-Daoust hypergeometric functions.

C. Analytic continuation of the Matsubara correlator

To obtain the retardated response function, we have to find the analytic continuation $i\omega_n \to \omega + i\epsilon$ of (32). We will first discuss the special case of $\eta = 1/2$, where the continuation is straightforward, leading to a simple picture of the behavior of the retardated response function. Then, we will turn to the more complicated case of $\eta > 1/2$, for which the calculations are more involved. We will see however that the simple picture of the case $\eta = 1/2$ is preserved.

1. *The case of* $\eta = 1/2$

In the case of $\eta = 1/2$, the analytic continuation is easily obtained from Eq. (27). We have for $u_j q < \omega < u_{j+1} q$:

$$\chi(q, \omega + i0) = 2\pi\alpha e^{i\pi \sum_{\ell=1}^{j} \eta_\ell} \prod_{j=1}^{N} |\omega^2 - (u_j q)^2|^{-\eta_j} \tag{35}$$

So that:

$$\mathfrak{Im}\chi(q, \omega \to u_j q + 0) \sim 2\pi\alpha \sin\left[\pi \sum_{l=1}^{j} \eta_l\right] |\omega^2 - (u_j q)^2|^{-\eta_j} \prod_{l\neq j} |(u_j^2 - u_l^2) q^2|^{-\eta_l}, \tag{36}$$

and:

$$\mathfrak{Im}\chi(q, \omega \to u_{j+1} q - 0) \sim 2\pi\alpha \sin\left[\pi \sum_{l=1}^{j} \eta_l\right] |\omega^2 - (u_{j+1} q)^2|^{-\eta_{j+1}} \prod_{l\neq j+1} |(u_{j+1}^2 - u_l^2) q^2|^{-\eta_l}, \tag{37}$$

showing that the spectral function has singularities with exponent $-\eta_j$ everytime $\omega \sim u_j q$. This result is is agreement with the result of Sec. II A for $\eta = 1/2$. The spectral function has a threshold for $\omega < u_1 q$ as there are no excitations of the system having a lower energy. Moreover, we note that for $j > 1$:

$$\frac{\mathfrak{Im}\chi(q, \omega \to u_j q + 0)}{\mathfrak{Im}\chi(q, \omega \to u_j q - 0)} = \frac{\sin\left[\pi \sum_{l=1}^{j} \eta_l\right]}{\sin\left[\pi \sum_{l=1}^{j-1} \eta_l\right]}, \tag{38}$$

which implies that the peaks of the spectral functions are asymmetric around $\omega = u_j q$. The imaginary part of the response function is represented on Fig. 1 for the case $\eta = 1/2$.

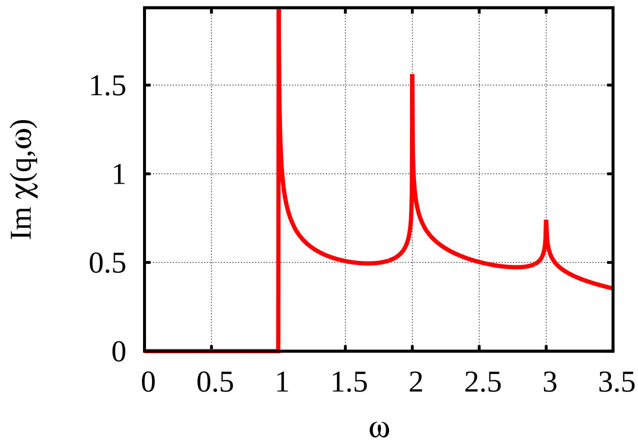

FIG. 1: The imaginary part of the response function $\chi(q,\omega)$ for the case $\eta_1 = 0.25$,$\eta_2 = 0.15$ and $\eta_3 = 0.1$. The velocities are $u_2 = 2u_1$ and $u_3 = 3u_1$. The unit of frequency ω is $u_1 q$. The unit of $\mathfrak{Im}\chi(q,\omega)$ is $2\pi\alpha/(u_1 q)$. Power-law divergences are obtained for $\omega = u_{1,2,3}q$.

2. The case of $\eta > 1/2$

In the general case, we need to find the analytic continuation of Eq. (32) for $\eta > 1/2$ or (34) for $\eta < 1/2$. Since the method is similar in the two cases, we will concentrate on the case of $\eta > 1/2$.

Formally, under the analytic continuation, the variables in Eq. (32) become:

$$\frac{u_N^2 - u_j^2}{u_N^2}\left(1 - \frac{\omega_n^2 t}{\omega_n^2 + u_N^2 q^2}\right) \to \left(1 - \frac{u_j^2}{u_N^2}\right)\left(1 + \frac{(\omega + i\epsilon)^2}{(u_N q)^2 - (\omega + i\epsilon)^2} t\right), \tag{39}$$

The Lauricella function F_D has cuts every time the real part of one of the variables is larger than one. In the case of $\omega > u_N q$, the real part of all the variables, according to (39) will remain less than one, and the analytic continuation is straightforward. We can then derive an equivalent for the Lauricella function in the limit of $\omega \to u_N q + 0$. We find:

$$\begin{aligned} &F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2}\left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right), \ldots, ; \frac{u_N^2 - u_{N-1}^2}{u_N^2}\left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right)\right) \\ &\sim \frac{\Gamma(\eta)\Gamma(\eta_N + 1/2 - \eta)}{\Gamma(1/2)\Gamma(\eta_N)}\left(\frac{\omega^2 - (u_N q)^2}{\omega^2 t}\right)^{\eta - \eta_N} \prod_{j=1}^{N-1}\left(1 - \frac{u_J^2}{u_N^2}\right)^{-\eta_j}, \end{aligned} \tag{40}$$

yielding:

$$\mathfrak{Im}\chi(q,\omega) = \frac{\pi 2^{1-\eta}\alpha^2 \Gamma(\eta_{N+1} - 2\eta)\sin(\pi\eta)}{\Gamma(\eta_N) u_N} \frac{(\omega^2 - (u_N q)^2)^{2\eta - \eta_N - 1}}{(\omega^2)^{\eta - \eta_N}\left(\frac{u_N^2}{\alpha^2}\right)^{\eta - 1}} \prod_{j=1}^{N-1}\left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j}. \tag{41}$$

In Eq. (41), the exponent predicted in Sec. II A is recovered. For $\omega \to u_N q - 0$, it is also possible to find an asymptotic estimation of the Lauricella function in the form:

$$\begin{aligned} &F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2}\left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right), \ldots, ; \frac{u_N^2 - u_{N-1}^2}{u_N^2}\left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right)\right) \\ &\sim \frac{\Gamma(\eta)\Gamma(\eta_N + 1/2 - \eta)}{\Gamma(1/2)\Gamma(\eta_N)}\left(\frac{(u_N q)^2 - \omega^2}{\omega^2 t}\right)^{\eta - \eta_N} e^{i\pi(\eta - \eta_N)\mathrm{sign}(\omega)} \prod_{j=1}^{N-1}\left(1 - \frac{u_J^2}{u_N^2}\right)^{-\eta_j}, \end{aligned} \tag{42}$$

giving:

$$\mathfrak{Im}\chi(q,\omega \to u_N q - 0) \sim \frac{\pi 2^{1-\eta}\alpha^2 \Gamma(\eta_{N+1} - 2\eta)\sin[\pi(\eta - \eta_N)]}{\Gamma(\eta_N) u_N} \frac{((u_N q)^2 - \omega^2)^{2\eta - \eta_N - 1}}{(\omega^2)^{\eta - \eta_N}\left(\frac{u_N^2}{\alpha^2}\right)^{\eta - 1}} \prod_{j=1}^{N-1}\left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j}, \tag{43}$$

i. e. the same power law divergence as in Eq. (41) but with a different prefactor. The ratio of the two expressions is: $\sin[\pi(\eta-\eta_N)]/\sin(\pi\eta)$, as previously noted in the special case of $\eta = 1/2$. For $\epsilon \to 0$, the imaginary part of (39) is positive for $\omega < u_N q$ and the real part of (39) is equal to one for $t = t_j = \frac{(u_N q/\omega)^2-1}{(u_N/u_j)^2-1}$. We have $t_1 < t_2 < \ldots < t_{N-1}$.

In particular, one finds that when $\omega < u_1 q$, $t_1 > 1$, so that no analytic continuation of the Lauricella function under the integral sign in (32) is needed. The response function remains purely real in that case, and the spectral function vanishes. In the case $u_j q < \omega < u_{j+1} q$, we find that $t_1 < \ldots < t_j < 1 < t_{j+1} < \ldots < t_{N-1}$. As a result, the integral has to be split into a sum of integrals over the intervals $[0, t_1]$, $[t_l, t_{l+1}]$ with $1 \le l \le j-1$ and $[t_j, 1]$. For each interval, the analytic continuation of the Lauricella function Eq. (B3) must be used in order to express the full integral. To give a concrete example of the procedure, we will at first focus on the case $N = 3$. We need to consider the following integral:

$$\mathrm{I}(q,\omega) = \int_0^1 dt t^{-\eta}(1-t)^{\eta-3/2} F_1\left(\frac{1}{2};\eta_1,\eta_2;\eta;\frac{u_3^2-u_1^2}{u_3^2}\left(1-\frac{\omega_n^2 t}{\omega_n^2+u_3^2q^2}\right);\frac{u_3^2-u_2^2}{u_3^2}\left(1-\frac{\omega_n^2 t}{\omega_n^2+u_3^2q^2}\right)\right). \quad (44)$$

For $\omega < u_1 q$, we have:

$$\mathrm{I}(q,\omega) = \int_0^1 dt t^{-\eta}(1-t)^{\eta-3/2} F_1\left(\frac{1}{2};\eta_1,\eta_2;\eta;\frac{u_3^2-u_1^2}{u_3^2}\left(1+\frac{\omega^2 t}{u_3^2q^2-\omega^2}\right);\frac{u_3^2-u_2^2}{u_3^2}\left(1+\frac{\omega^2 t}{u_3^2q^2-\omega^2}\right)\right). \quad (45)$$

Let us now consider the case of $u_1 q < \omega < u_2 q$. First, we split the integral with the rule given above and consider the intervals $t \in [0, \frac{\frac{(u_3 q)^2}{\omega^2}-1}{\frac{u_3^2}{u_1^2}-1}]$ and $t \in [\frac{\frac{(u_3 q)^2}{\omega^2}-1}{\frac{u_3^2}{u_1^2}-1}, 1]$. In the first interval, the analytic continuation is straightforward. In the second integral, we must use (A4).

To calculate the integrals, it is convenient to perform the following change of variables in the first and second interval respectively:

$$\begin{aligned} t &= \frac{\frac{u_3 q}{\omega}-1}{\frac{u_3}{u_1}-1} s_1 \\ t &= \frac{\frac{u_3 q}{\omega}-1}{\frac{u_3}{u_1}-1} + \left(1-\frac{\frac{u_3 q}{\omega}-1}{\frac{u_3}{u_1}-1}\right) s_2 \end{aligned}$$

In the integration over s_1, we don't need an analytic continuation of the Appell function. In the s_2 integration, one of the two variables is larger than 1 and we find the analytic continuation using Eq. (A4).

Since we are interested in calculating the imaginary part of (44), only the term proportional to $e^{i\pi\eta_1}$ gives a contribution and we can perform explicitly the integral when $|\omega| \to u_1 q$. The final results for the imaginary part of the integral (44) yields:

$$\begin{aligned} \mathfrak{Im}\mathrm{I}(\omega,q) = {} & \frac{\pi\Gamma(\eta)}{\Gamma(1/2)\Gamma(1/2+\eta-\eta_1)} \frac{\left(\frac{u_1^2}{u_3^2}\right)^{\eta-1/2-\eta_1}}{\left(\frac{u_3^2-u_1^2}{u_3^2}\right)^{\eta-\eta_2-1}\left(\frac{u_2^2-u_1^2}{u_3^2}\right)^{\eta_2}} \left(\frac{\frac{\omega^2}{u_1^2}-q^2}{q^2-\frac{\omega^2}{u_3^2}}\right)^{2\eta-1-\eta_1} \left(\frac{\frac{u_3^2}{u_1^2}-1}{\frac{u_3^2q^2}{\omega^2}-1}\right)^{1/2} \\ & \times \int_0^1 \frac{ds s^{\eta-\eta_1-1/2}(1-s)^{\eta-3/2}}{\left(1+\frac{\frac{\omega^2}{u_1^2}-q^2}{q^2-\frac{\omega^2}{u_3^2}}s\right)^{\eta}\left(1+\frac{\omega^2-u_1^2q^2}{u_3^2q^2-\omega^2}s\right)^{\eta-1}} F_1\left(1-\eta_1;\frac{1}{2},\eta_2;\eta+\frac{1}{2}-\eta_1;-\frac{\omega^2-u_1^2q^2}{u_3^2q^2-\omega^2}s;\frac{u_3^2-u_2^2}{u_2^2-u_1^2}\frac{\omega^2-u_1^2q^2}{u_3^2q^2-\omega^2}s\right) \end{aligned} \quad (46)$$

When $\omega \to u_1 q$, the integral is behaving as $(\omega^2 - u_1^2 q^2)^{2\eta-1-\eta_1}$, in agreement with (22). For $\omega \to u_2 q$, we need to consider the behavior of the Appell function as one of its arguments is going to unity, while the other is negative. Using the results from App. C, we find that when $\eta_1 < 1/2$, we can use Eq. (C8) to approximate Eq. (46) as:

$$\begin{aligned} \mathfrak{Im}I(\omega,q) &\propto \int_0^1 ds s^{\eta-3/2}\left(\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\frac{(u_2q)^2-\omega^2}{(u_3q)^2-\omega^2}-s\right)^{\eta-\eta_2-1/2} \\ &\propto \left(\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\frac{(u_2q)^2-\omega^2}{(u_3q)^2-\omega^2}\right)^{2\eta-\eta_2-1} \end{aligned} \quad (47)$$

Again, this result is in agreement with the power law divergence expected from (22).

We now turn to the case $u_2q < \omega < u_3q$. In that case, we have to split the integral in Eq. (44) into three integrations. The first one, on $[0, \frac{(u_3q/\omega)^2-1}{(u_3/u_1)^2-1}]$ does not contribute to the imaginary part. The second one, on the interval $[\frac{(u_3q/\omega)^2-1}{(u_3/u_1)^2-1}, \frac{(u_3q/\omega)^2-1}{(u_3/u_2)^2-1}]$, requires an analytic continuation of the F_1 function using (A4) and gives a contribution to the imaginary part:

$$\begin{aligned}&\frac{\pi\Gamma(\eta)(1-(u_1/u_3)^2)^{1/2}}{\Gamma(1/2)\Gamma(\eta_1)\Gamma(1/2+\eta-\eta_1)}\left(\frac{u_3^2}{u_1^2}\right)^{2-\eta}\left(\frac{u_2^2-u_1^2}{u_3^2-u_2^2}\right)^{\eta-\eta_1+1/2}\left(\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\right)^{\eta_2}\left(\frac{(u_3q)^2-\omega^2}{\omega^2-(u_1q)^2}\right)^{\eta-1}\left(1-\frac{u_1^2q^2}{\omega^2}\right)^{-1/2}\\&\times\int_0^1 ds s^{\eta-\eta_1-1/2}\left(1+\frac{u_2^2-u_1^2}{u_3^2-u_2^2}s\right)^{1-\eta}\left(1+\frac{u_3^2}{u_1^2}\frac{u_2^2-u_1^2}{u_3^2-u_2^2}s\right)^{-\eta}\left(1-\frac{u_2^2-u_1^2}{u_3^2-u_2^2}\frac{(u_3q)^2-\omega^2}{\omega^2-(u_1q)^2}s\right)^{\eta-3/2}\\&\times F_1\left(1-\eta_1;\frac{1}{2},\eta_2;\eta+\frac{1}{2}-\eta_1;-\frac{u_2^2-u_1^2}{u_3^2-u_2^2}s,s\right)\end{aligned} \tag{48}$$

The last integration, on $[\frac{(u_3q/\omega)^2-1}{(u_3/u_2)^2-1}, 1]$, requires an analytic continuation of F_1 using (A7), and contributes two terms. The first one is:

$$\begin{aligned}&\frac{\pi\Gamma(\eta)\Gamma(1-\eta_2)|\omega|(u_3^2)^{2\eta-\eta_1-1}(u_2^2)^{-\eta}(u_1^2)^{\eta_1+1/2-\eta}}{\Gamma(1/2)\Gamma(\eta_1)\Gamma(\eta-1/2)\Gamma(1-\eta_1-\eta_2)}\frac{(u_2^2-u_1^2)^{\eta+1/2-2\eta_1-\eta_2}}{(u_3^2-u_2^2)^{1-\eta_1-\eta_2}(u_3^2-u_1^2)^{\eta-\eta_1-1}}\frac{(\omega^2-(u_2q)^2)^{\eta-1/2}}{((u_3q)^2-\omega^2)^{\eta}}\\&\times\int_0^1 ds(1-s)^{\eta-3/2}\left(1+\frac{u_3^2}{u_2^2}\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)^{-\eta}\left(1+\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)^{1-\eta}\left(1+\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)^{\eta-\eta_1-1/2}\\&F_1\left(1-\eta_1;\frac{1}{2},\frac{3}{2}-\eta;2-\eta_1-\eta_2;\frac{u_1^2-u_2^2}{u_3^2-u_2^2},\frac{1}{1+\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s}\right),\end{aligned} \tag{49}$$

and for $\eta-\eta_2-1/2<0$ behaves as $(\omega^2-(u_2q)^2)^{2\eta-\eta_2-1}$ in agreement with (22). The second one is:

$$\begin{aligned}&\frac{\Gamma(\eta)\Gamma(1-\eta_2)\sin[\pi(\eta_1+\eta_2)]}{\Gamma(1/2)\Gamma(1/2+\eta-\eta_2)}\left(\frac{u_3^2-u_2^2}{u_2^2-u_1^2}\right)^{\eta_1}\left(\frac{\omega^2}{u_2^2}\frac{u_3^2-u_2^2}{(u_3q)^2-\omega^2}\right)^{\eta}\left(\frac{u_3^2}{\omega^2}\frac{\omega^2-(u_2q)^2}{u_3^2-u_2^2}\right)^{\eta-1/2}\left(\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}\right)^{\eta-\eta_2-1/2}\\&\times\int_0^1 s^{\eta-\eta_2-1/2}(1-s)^{\eta-3/2}\left(1+\frac{u_3^2}{u_2^2}\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)^{-\eta}\left(1+\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)^{-1/2}\\&\times F_1\left(1-\eta_2;\frac{1}{2},\eta_1;\eta+\frac{1}{2}-\eta_2;-\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s,-\frac{u_3^2-u_1^2}{u_2^2-u_1^2}\frac{\omega^2-(u_2q)^2}{(u_3q)^2-\omega^2}s\right)\end{aligned} \tag{50}$$

The latter term contributes a divergence $(\omega^2-(u_2q)^2)^{2\eta-\eta_2-1}$ as $\omega\to u_2q+0$ in agreement with (22).

To summarize, the qualitative behavior of the spectral function is the same as in the special case of $\eta=1/2$. The spectral function has a threshold at $\omega=u_1q$, and has power law singularities for $\omega=u_jq$ with an exponent given by Eq. (22).

The case of a general N is treated in the Appendix D. It is found that the leading singularity for $\omega\to u_jq$ is again a power law divergence, with exponent given by (22).

III. CONCLUSION

We derived a analytical expressions for the zero temperature Fourier transform of the density-density correlation function and the bosonic Green's function of a multicomponent Luttinger liquid with different velocities. By using both a Schwinger identity and a generalized Feynman identity, we derived exact integral representations while an approximate analytical form was given for frequencies close to the characteristic frequencies of the different collective modes of the system. We derived in detail the analytic continuation for generic N and discussed, as an example, the case $N=3$. Power-law singularities are found every time the frequency is equal to the characteristic frequency of a collective mode ($\omega_j(q)\sim u_jq$), with the same exponent $2\eta-\eta_j-1$, but a different weight when approaching the singularity from the left or from the right. The power-law singularity replaces the expected delta function for noninteracting particles. Moreover if the characteristic exponent at the singularity becomes negative, as in the case of systems with attractive interaction when considering density-density correlations or in systems with repulsion when considering bosonic Green's functions, a cusp is expected to replace the power-law divergence. All these results are

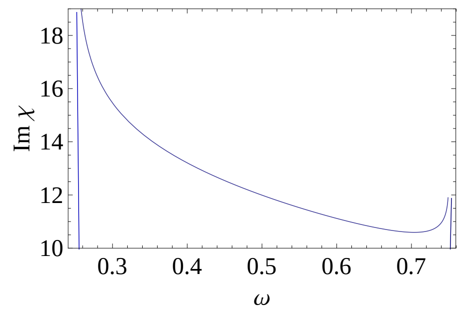

FIG. 2: Spectral function close to the singularity for N=2 and the values of the parameters: $\eta_1 = 0.3, \eta_2 = 0.1, \eta = 0.6, u_1 = 0.25, u_2 = 0.75$.

valid in the ground state. For nonzero temperature T, the power-law divergence at $\omega = u_j q$ is replaced by a maximum diverging as $T^{2\eta-\eta_j-1}$ as $T \to 0$. In the vicinity of the maximum, from simple scaling, we expect $\mathrm{Im}\chi(q,\omega) \sim (k_B T)^{2\eta-\eta_j-1} f_{\text{temp.}}[(\omega - u_j q)/k_B T]$, with $f_{\text{temp.}}(x \gg 1) \sim x^{2\eta-\eta_j-1}$. If we now turn to a zero temperature system of finite length L, we expect the power-law divergence at $\omega = u_j q$ to be replaced by a maximum diverging as $(1/L)^{2\eta-\eta_j-1}$, and in the vicinity of the maximum $\mathrm{Im}\chi(q,\omega) \sim (1/L)^{2\eta-\eta_j-1} f_{\text{len.}}[L(\omega/u_j - q)]$ with $f_{\text{len.}} \sim (x \gg 1) \sim x^{2\eta-\eta_j-1}$. If the calculation of both functions $f_{\text{len.}}$ and $f_{\text{temp.}}$ remains an open problem, the scaling arguments suggest that the power law behavior of $\mathrm{Im}\chi(q,\omega)$ is observable for finite temperature and finite size provided $|\omega - u_j q| \gg k_B T, u_j/L$. Such behavior could be probed by Bragg or time of flight spectroscopy in the case of atomic gases of mixed species or by inelastic neutron scattering technique in one quantum magnets with orbital and spin modes. An obvious extension of the results of our manuscript is the calculation of fermion (or anyon) spectral functions in multicomponent Luttinger liquid. In the case of a two-component liquid, the fermion spectral functions are expressible in terms of Appell hypergeometric functions[74]. For the case of three or more components, the results of our manuscript hint that the fermion spectral functions should be expressible as Srivastava-Daoust hypergeometric functions or a suitable generalization.

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Appendix A: Analytic continuation of the Appell F_1 function

In the calculation of the response function, an analytic continuation of the Lauricella hypergeometric function $F_D^{(N-1)}$ is necessary. In the present appendix, we present the analytic continuation of the Appell F_1 hypergeometric function that corresponds to the particular case of $N = 3$. The analysis of that case is the stepping stone for the case of general N.

In order to find the analytic continuation, we start from the integral representation of the Appell hypergeometric function[67,68]

$$F_1(a; b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tz_1)^{b_1}(1-tz_2)^{b_2}} \tag{A1}$$

when $|z_1|, |z_2| < 1$, expanding in series (A1) and integrating w.r. t. t gives back the series expansion. We wish to use (A1) to express $\lim_{\epsilon_1,\epsilon_2 \to 0_+} F_1(a; b_1, b_2; c; x_1 + i\epsilon_1, x_2 + i\epsilon_2)$ for x_1 and x_2 real for the cases $x_2 < 1 < x_1$ and $1 < x_2 < x_1$.

1. case $x_2 < 1 < x_1$

In order to calculate the integral (A1), we need to take into account the sole branch cut of $(1 - t(x_1 + i\epsilon_1))^{b_1}$ for $t > 1/x_1$. Because of this cut, we have for $t > x_1$, $(1 - t(x_1 + i\epsilon_1))^{b_1} = e^{-i\pi b_1}(tx_1 - 1)^{b_1}$. Therefore, we split the t

integral in (A1) into two integrations on $[0,1/x_1]$ and $[1/x_1,1]$. After a change of variable $t=s/x_1$, we find for the $[0,x_1]$ integral :

$$\int_0^{1/x_1} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx_1)^{b_1}(1-tx_2)^{b_2}} = \frac{1}{x_1^a}\frac{\Gamma(a)\Gamma(1-b_1)}{\Gamma(1+a-b_1)} F_1(a;a+1-c,b_2;a+1-b_1;1/x_1;x_2/x_1), \tag{A2}$$

a purely real expression. Then, for the $[1/x_1,1]$ integration, we find:

$$\begin{aligned}\int_{1/x_1}^{1} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1+i0_+))^{b_1}(1-tx_2)^{b_2}} &= e^{i\pi b_1}\frac{(x_1-1)^{c-a-b_1}}{x_1^{c-b_2-1}(x_1-x_2)^{b_2}}\frac{\Gamma(c-a)\Gamma(1-b_1)}{\Gamma(1+c-a-b_1)} \\ &\times F_1\left(1-b_1;1-a;b_2;c-a+1-b_1;1-x_1,\frac{x_2(x_1-1)}{x_1-x_2}\right),\end{aligned} \tag{A3}$$

where we have used the linear change of variables $t=1/x_1+s(1-1/x_1)$. Our result for the analytic continuation is then:

$$\begin{aligned}F_1(a;b_1,b_2;c;x_1+i0_+,x_2) &= \frac{\Gamma(c)\Gamma(1-b_1)}{\Gamma(c-a)\Gamma(1+a-b_1)}\frac{1}{x_1^a}F_1(a;a+1-c,b_2;a+1-b_1;1/x_1;x_2/x_1) \\ &+e^{i\pi b_1}\frac{\Gamma(c)\Gamma(1-b_1)}{\Gamma(a)\Gamma(1+c-a-b_1)}\frac{(x_1-1)^{c-a-b_1}}{x_1^{c-b_2-1}(x_1-x_2)^{b_2}}F_1\left(1-b_1;1-a;b_2;c-a+1-b_1;1-x_1,\frac{x_2(x_1-1)}{x_1-x_2}\right).\end{aligned} \tag{A4}$$

2. case $1<x_2<x_1$

In that case, we need to consider both the branch cut of $(1-t(x_1+i0_+))^{b_1}$ and of $(1-t(x_2+i0_+))^{b_2}$. Thus, we are lead to split the integral (A1) into three integrations over $[0,1/x_1]$, $[1/x_1,1/x_2]$ and $[1/x_2,1]$. The first of these integrals is still given by (A2). The second integral is given by:

$$\begin{aligned}\int_{1/x_1}^{1/x_2} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1+i0_+))^{b_1}(1-tx_2)^{b_2}} &= e^{i\pi b_1}\frac{\Gamma(1-b_1)\Gamma(1-b_2)}{\Gamma(2-b_1-b_2)}\frac{(x_1-x_2)^{1-b_1-b_2}(x_1-1)^{c-a-1}}{x_1^{c-b_2-1}x_2^{1-b_1}} \\ &\times F_1\left(1-b_1;1-a,1+a-c;2-b_1-b_2;1-\frac{x_1}{x_2},\frac{x_1-x_2}{x_2(x_1-1)}\right),\end{aligned} \tag{A5}$$

where we have used the change of variable $t=1/x_1+s(1/x_2-1/x_1)$ and taken into account the branch cut of $(1-t(x_1+i0_+))^{b_1}$. For the third integral, we have:

$$\begin{aligned}\int_{1/x_2}^{1} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1+i0_+))^{b_1}(1-t(x_2+i0_+))^{b_2}} &= e^{i\pi(b_1+b_2)}\frac{\Gamma(1-b_2)\Gamma(c-a)}{\Gamma(1+c-a-b_2)}\frac{(x_2-1)^{c-a-b_2}}{x_2^{a-b_1}(x_1-x_2)^{b_1}} \\ &\times F_1\left(1-b_2;1-a,b_1;1+c-a-b_2;1-x_2,\frac{x_1(1-x_2)}{x_1-x_2}\right),\end{aligned} \tag{A6}$$

where we have taken both branch cuts into account, and we have used the change of variables $t=1/x_2+(1-1/x_2)s$. The final result is:

$$\begin{aligned}F_1(a,b_1,b_2;c;x_1+i0_+,x_2+i0_+) &= \frac{\Gamma(c)\Gamma(1-b_1)}{\Gamma(c-a)\Gamma(1+a-b_1)}\frac{1}{x_1^a}F_1\left(a;1+a-c,b_2;1+a-b_1;\frac{1}{x_1},\frac{x_2}{x_1}\right) \\ &+e^{i\pi b_1}\frac{\Gamma(c)\Gamma(1-b_1)\Gamma(1-b_2)}{\Gamma(a)\Gamma(c-a)\Gamma(2-b_1-b_2)}\frac{(x_1-x_2)^{1-b_1-b_2}(x_1-1)^{c-a-1}}{x_1^{c-b_2-1}x_2^{1-b_1}}F_1\left(1-b_1;1-a,1+a-c;2-b_1-b_2;1-\frac{x_1}{x_2},\frac{x_1-x_2}{x_2(x_1-1)}\right) \\ &+e^{i\pi(b_1+b_2)}\frac{\Gamma(c)\Gamma(1-b_2)}{\Gamma(a)\Gamma(1+c-a-b_2)}\frac{(x_2-1)^{c-a-b_2}}{x_2^{a-b_1}(x_1-x_2)^{b_1}}F_1\left(1-b_2;1-a,b_1;1+c-a-b_2;1-x_2,\frac{x_1(1-x_2)}{x_1-x_2}\right)\end{aligned} \tag{A7}$$

For the case $N=3$, the expressions (A4) and (A7) must be injected in the integral (32) after analytic continuation to yield the response function.

Appendix B: Analytic continuation of the Lauricella F_D function

The Lauricella F_D function has the integral representation:

$$F_D(a;b_1,\ldots,b_n;c;z_1,\ldots,z_n)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 dt\frac{t^{a-1}(1-t)^{c-a-1}}{\prod_{j=1}^n(1-z_jt)^{b_j}} \tag{B1}$$

If we want to calculate for $x_n<x_{n-1}<\ldots<x_2<x_1$:

$$\lim_{\epsilon_j\to 0_+} F_D(a;b_1,\ldots,b_n;c;x_1+i\epsilon_1,\ldots,x_n+i\epsilon_n) \tag{B2}$$

we have to consider the cuts of the functions $(1-tx_j+i0_-)^{-b_j}$. We are thus led to consider separately n cases, i. e. $x_n<\ldots<x_{j+1}<1<x_j<\ldots<x_1$ with $j=1,\ldots,n-1$ and $1<x_n<\ldots<x_1$.

In the case of $x_n<\ldots<x_{j+1}<1<x_j<\ldots<x_1$, the integral (B1) has to be split into $j+1$ integrations over the intervals $[0,1/x_1]$,$[1/x_1,1/x_2]$,…,$[1/x_j,1]$. In analogy to the case of the Appell F_1 function, each integral gives a contribution proportional to a Lauricella F_D function. The resulting expression is:

$$\begin{aligned}
&F_D^{(N)}(a;b_1,\ldots,b_N;c;x_1+i0,\ldots,x_N+i0)=\\
&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\left[\frac{\Gamma(a)\Gamma(1-b_1)}{\Gamma(1+a-b_1)x_1^a}F_D^{(N)}\left(a;1+a-c,b_2,\ldots,b_N;1+a-b_1;\frac{1}{x_1},\frac{x_2}{x_1}\ldots,\frac{x_N}{x_1}\right)\right.\\
&+\sum_{m=1}^{j-1}\frac{\Gamma(1-b_m)\Gamma(1-b_{m+1})}{\Gamma(2-b_m-b_{m+1})}\frac{(x_m-x_{m+1})^{1-b_m-b_{m+1}}(1-x_m)^{c-a-1}}{x_m^{c-1-b_{m+1}}x_{m+1}^{1-b_m}}\prod_{l\neq m,m+1}\left|\frac{x_l-x_m}{x_m}\right|^{-b_l}e^{i\pi\sum_{l=1}^m b_l}\times\\
&F_D^{(N)}\left(1-b_m;b_1,\ldots,b_{m-1},1-a,1+a-c,b_{m+2},\ldots,b_N;2-b_m-b_{m+1};\frac{1-\frac{x_m}{x_{m+1}}}{1-\frac{x_m}{x_1}},\ldots,\frac{1-\frac{x_m}{x_{m+1}}}{1-\frac{x_m}{x_{m-1}}},1-\frac{x_m}{x_{m+1}},\frac{1-\frac{x_m}{x_{m+1}}}{1-x_m},\right.\\
&\left.\frac{1-\frac{x_m}{x_{m+1}}}{1-\frac{x_m}{x_{m+2}}},\ldots,\frac{1-\frac{x_m}{x_{m+1}}}{1-\frac{x_m}{x_N}}\right)\\
&+\frac{\Gamma(1-b_j)\Gamma(c-a)}{\Gamma(1+c-a-b_j)}\frac{(x_j-1)^{c-a-b_j}}{x_j^{c-1}}\prod_{l\neq j}\left|\frac{x_l-x_j}{x_j}\right|^{-b_l}e^{i\pi\sum_{l=1}^j b_l}\times\\
&\left.F_D^{(N)}\left(1-b_j;b_1,\ldots,b_{j-1},1-a,b_{j+1},\ldots,b_N;1+c-a-b_j;\frac{1-x_j}{1-\frac{x_j}{x_1}},\ldots,\frac{1-x_j}{1-\frac{x_j}{x_{j-1}}},1-x_j,\frac{1-x_j}{1-\frac{x_j}{x_{j+1}}},\ldots,\frac{1-x_j}{1-\frac{x_j}{x_N}}\right)\right]
\end{aligned} \tag{B3}$$

By reducing to $N=2$ it can be checked that the results of Sec. A are recovered.

In the case $1<x_1<\ldots<x_n$, we have to split the integral (B1) into $n+1$ integrations over the intervals $[0,1/x_1]$,…,$[1/x_j,1/x_{j+1}]$,…,$[1/x_n,1]$. Each integration contributes a term proportional to a Lauricella F_D function.

Appendix C: Asymptotic expansion of the Appell F_1 function

We wish to obtain an asymptotic expansion of the Appell function $F_1(a;b_1,b_2;c;x_1,x_2)$ in the case of $x_2<0$ and $x_1\to 1_-$. First, we need to obtain an expression of the Appell function in the form of a convergent series for all $x_2<0$. We consider the series expansion for the Appell hypergeometric function F_1:

$$F_1(a;b_1,b_2;c;x_1,x_2)=\sum_{n_1,n_2}\frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c)_{n_1+n_2}}\frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}, \tag{C1}$$

$$=\sum_{n_1}\frac{(a)_{n_1}(b_1)_{n_1}}{(c)_{n_1}}\frac{x_1^{n_1}}{n_1!}{}_2F_1(a+n_1,b_2;c+n_1;x_2), \tag{C2}$$

where we have used the notation[71]:

$$(a)_n=\frac{\Gamma(n+a)}{\Gamma(a)} \tag{C3}$$

Using the second line of (C1), we can define the function F_1 for all $x_2 \notin [1, +\infty[$. For $x_2 < 0$, we can use Eq. (15.3.4) from Ref. 71, to rewrite:

$$\begin{aligned} F_1(a; b_1, b_2; c; x_1, x_2) &= (1-x_2)^{-b_2} \sum_{n_1} \frac{(a)_{n_1}(b_1)_{n_1}}{(c)_{n_1}} \frac{x_1^{n_1}}{n_1!} {}_2F_1\left(b_2, c-a; c+n_1; \frac{x_2}{x_2-1}\right), \\ &= (1-x_2)^{-b_2} \sum_{n_1,n_2} \frac{(a)_{n_1}(b_1)_{n_1}(b_2)_{n_2}(c-a)_{n_2}}{(c)_{n_1+n_2}} \frac{x_1^{n_1}}{n_1!}\frac{1}{n_2!}\left(\frac{x_2}{x_2-1}\right)^{n_2}, \\ &= (1-x_2)^{-b_2} F_3\left(a, c-a; b_1, b_2; c; x_1, \frac{x_2}{x_2-1}\right), \qquad \text{(C4)} \\ &= (1-x_2)^{-b_2} \sum_{n_2} \frac{(c-a)_{n_2}(b_2)_{n_2}}{(c)_{n_2} n_2!}\left(\frac{x_2}{x_2-1}\right)^{n_2} {}_2F_1(a, b_1; c+m; x_1) \qquad \text{(C5)} \end{aligned}$$

where we have used the convergence of the Gauss hypergeometric series to obtain the last two lines. with this convergent series, we can analyze its behavior as $x_1 \to 1_-$. First, when $c+m-a-b_1 > 0$, we have from Eq. (15.1.20) in Ref.71:

$$\lim_{x\to 1_-} {}_2F_1(a, b_1; c+m; x_1) = \frac{\Gamma(c+m)\Gamma(c+m-a-b_1)}{\Gamma(c+m-a)\Gamma(c+m-b_1)}. \tag{C6}$$

When $c+m-a-b_1 < 0$, using first Eq. (15.3.3) in Ref.71, we find that as $x \to 1_-$,

$${}_2F_1(a, b_1; c+m; x_1) \sim (1-x_1)^{c-a-b_1+m} \frac{\Gamma(c+m)\Gamma(a+b_1-c-m)}{\Gamma(a)\Gamma(b_1)} \tag{C7}$$

Thus, when $c-a-b_1 < 0$, we have that:

$$F_1(a; b_1; b_2; c; x_1, x_2) \sim (1-x_2)^{-b_2} \frac{\Gamma(c)\Gamma(a+b_1-c)}{\Gamma(a)\Gamma(b_1)} (1-x_1)^{c-a-b_1}, \tag{C8}$$

while for $c-a-b_1 > 0$,

$$\lim_{x_1\to 1_-} F_1(a; b_1; b_2; c; x_1, x_2) = (1-x_2)^{-b_2} \frac{\Gamma(c)\Gamma(c+a-b_1)}{\Gamma(c-a)\Gamma(c-b_1)} {}_2F_1\left(c-a-b_1, b_2; c-b_1; \frac{x_2}{x_2-1}\right) \tag{C9}$$

Let us now return to the case of a general N. We have to consider the integral:

$$I^{(N-1)}(q,\omega) = \int_0^1 dt t^{-\eta}(1-t)^{\eta-3/2} F_D^{(N-1)}\left(\frac{1}{2}; \{\eta_j\}_{1\leq j\leq N-1}; \eta; \left\{\frac{u_N^2-u_j^2}{u_N^2}\left(1+\frac{t}{\frac{u_N^2 q^2}{(\omega+i0)^2}-1}\right)\right\}_{1\leq j\leq N-1}\right) \tag{C10}$$

Appendix D: Analytic continuation of the Matsubara correlator for generic η and N components

With the help of Eq. (B3) we can write for $t_l < t < t_{l+1}$:

$$F_D^{(N-1)}\left(\frac{1}{2}; \{\eta_j\}_{1\leq j\leq N-1}; \eta; \left\{\frac{u_N^2-u_j^2}{u_N^2}\left(1+\frac{t}{\frac{u_N^2 q^2}{(\omega+i0)^2}-1}\right)\right\}_{1\leq j\leq N-1}\right) = \sum_{m=0}^{l-1} \varphi_m(t) + \psi_l(t), \tag{D1}$$

where:

$$\begin{aligned} \varphi_0(t) = &\frac{\Gamma(\eta)\Gamma(1-\eta_1)}{\Gamma(\eta-1/2)\Gamma(3/2-\eta_1)} \frac{1}{\left[\left(1-\frac{u_1^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{(\omega+i0)^2}-1}\right)\right]^{1/2}} \times \\ &F_D^{(N-1)}\left(\frac{1}{2}; \frac{3}{2}-\eta, \{\eta_j\}_{2\leq j\leq N-1}; \frac{3}{2}-\eta_1; \frac{1}{\left(1-\frac{u_1^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)}, \left\{\frac{u_N^2-u_j^2}{u_N^2-u_1^2}\right\}_{2\leq j\leq N-1}\right), \end{aligned} \tag{D2}$$

for $m \geq 1$:

$$\varphi_m(t) = \frac{\Gamma(\eta)\Gamma(1-\eta_m)\Gamma(1-\eta_{m+1})}{\Gamma(1/2)\Gamma(\eta-1/2)\Gamma(2-\eta_m--\eta_{m+1})}\left(\frac{u_{m+1}^2-u_m^2}{u_N^2-u_{m+1}^2}\right)^{1-\eta_m}\left(\frac{u_{m+1}^2-u_m^2}{u_N^2-u_m^2}\right)^{-\eta_{m+1}}\prod_{k\neq m,m+1}\left|\frac{u_m^2-u_k^2}{u_N^2-u_m^2}\right|^{-\eta_k} e^{i\pi\sum_1^m \eta_k}$$
$$\frac{\left[1-\left(1-\frac{u_m^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right]^{\eta-3/2}}{\left[\left(1-\frac{u_m^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right]^{\eta-1}} F_D^{(N-1)}\left(1-\eta_m,\{\eta_k\}_{1\leq k\leq m-1},\frac{1}{2},\frac{1}{2}-\eta,\{\eta_k\}_{m+2\leq k\leq N-1};2-\eta_m-\eta_{m+1};\right. \tag{D3}$$
$$\left.\left\{\frac{(u_m^2-u_{m+1}^2)(u_N^2-u_k^2)}{(u_m^2-u_k^2)(u_N^2-u_{m+1}^2)}\right\}_{1\leq k\leq m-1},\frac{(u_m^2-u_{m+1}^2)}{(u_N^2-u_{m+1}^2)},\frac{\frac{(u_{m+1}^2-u_m^2)}{(u_N^2-u_{m+1}^2)}}{\left(1-\frac{u_m^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)},\left\{\frac{(u_m^2-u_{m+1}^2)(u_N^2-u_k^2)}{(u_m^2-u_k^2)(u_N^2-u_{m+1}^2)}\right\}_{m+2\leq k\leq N-1}\right)$$

and:

$$\psi_l(t) = \frac{\Gamma(1-\eta_l)\Gamma(\eta-1/2)}{\Gamma(\eta-\eta_l+1/2)}\prod_{k\neq l}\left|\frac{u_l^2-u_k^2}{u_N^2-u_k^2}\right|^{-\eta_k} e^{i\pi\sum_1^l \eta_k}\frac{\left[\left(1-\frac{u_l^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)-1\right]^{\eta-\eta_l-1/2}}{\left[\left(1-\frac{u_m^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right]^{\eta-1}}\times \tag{D4}$$
$$F_D^{(N-1)}\left(1-\eta_l,\{\eta_k\}_{1\leq k\leq l-1},\frac{1}{2},\{\eta_k\}_{l+1\leq k\leq N-1};\frac{1}{2}+\eta-\eta_l;\left\{\frac{u_N^2-u_k^2}{u_N^2-u_l^2}\left(1-\left(1-\frac{u_l^2}{u_N^2}\right)\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right)\right\}_{1\leq k\leq N-1}\right)$$

So that for $u_jq < \omega < u_{j+1}q$:

$$I^{(N-1)}(q,\omega) = \sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}} dt t^{-\eta}(1-t)^{\eta-3/2}\left(\sum_{m=0}^{l-1}\varphi_m(t)+\psi_l(t)\right)+\int_{t_j}^{1} dt t^{-\eta}(1-t)^{\eta-3/2}\left(\sum_{m=0}^{j-1}\varphi_m(t)+\psi_j(t)\right)$$
$$+\int_0^{t_1} dt t^{-\eta}(1-t)^{\eta-3/2}F_D^{(N-1)}\left(\frac{1}{2};\{\eta_j\}_{1\leq j\leq N-1};\eta;\left\{\frac{u_N^2-u_j^2}{u_N^2}\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right\}_{1\leq j\leq N-1}\right) \tag{D5}$$

We can rearrange that sum into:

$$I^{(N-1)}(q,\omega) = \sum_{m=0}^{j-1}\int_{t_{m+1}}^{1} dt t^{-\eta}(1-t)^{\eta-3/2}\varphi_m(t)+\sum_{l=1}^{j-1}\int_{t_l}^{t_{l+1}} dt t^{-\eta}(1-t)^{\eta-3/2}\psi_l(t)+\int_{t_j}^{1} dt t^{-\eta}(1-t)^{\eta-3/2}\psi_j(t)$$
$$+\int_0^{t_1} dt t^{-\eta}(1-t)^{\eta-3/2}F_D^{(N-1)}\left(\frac{1}{2};\{\eta_j\}_{1\leq j\leq N-1};\eta;\left\{\frac{u_N^2-u_j^2}{u_N^2}\left(1+\frac{t}{\frac{u_N^2q^2}{\omega^2}-1}\right)\right\}_{1\leq j\leq N-1}\right) \tag{D6}$$

So we have to calculate $2j$ integrals. When in the last integral we do substitute the expression of (D4), a factor $\left(\omega^2-(u_jq)^2\right)^{2\eta-\eta_j-1}$ appears with an integral which is regular in the limit $\omega \to u_jq$, so the asymptotic behavior previously predicted is recovered. The first class of integrals in (D6) can be, instead, manipulated by using the definition (D3) and performing the following change of variables:

$$t = \frac{\left(\frac{(u_Nq)^2}{\omega^2}-1\right)}{\left(\frac{(u_N)^2}{\omega_{m+1}^2}-1\right)}+\left(1-\frac{\left(\frac{(u_Nq)^2}{\omega^2}-1\right)}{\left(\frac{(u_N)^2}{\omega_{m+1}^2}-1\right)}\right)s. \tag{D7}$$

The integrals turn out to be regular for $\omega > u_jq$ and for $\omega \to u_jq+0$ they behave as $\left(\omega^2-(u_jq)^2\right)^{\eta-1/2}$ giving only

a subdominant contribution.

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